Energy fields of partially coherent sources

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Random acoustic fields and their energetic quantities (acoustic active and reactive intensities, potential and kinetic energy densities) are described in terms of the mutual coherences between sources. Conditions to correctly construct the coherence matrix of sources in a multivariate random process are given. It is shown that the description of a sound field using the coherence matrix of source is equivalent to the superposition of a number of independent coherent fields, which do not correspond to the original localized sources. A method based on processing the principal components of the coherence matrix of sources is given to reduce the number of necessary fields. The coherence function between acoustic pressure and particle velocity and the curl of active intensity are proposed as two indicators for estimating the degree of coherence and the polarization of acoustic fields. These indicators are analyzed theoretically and experimentally. The description of the structure of partially coherent fields is generalized by the definition of the field matrix whose rank is an indicator of the local complexity of an acoustic field. © 1998 Acoustical Society of America. [S0001-4966(98)04102-2]

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INTRODUCTION

For a better comprehension of the characteristics of sound intensity fields observed around real sources, models consisting of point sources have often been used to simulate the complexity of interference fields.1–4 It has been shown that the influence of the coherence between sources on the directivity of acoustic radiation is considerable.3 The relationship of the coherence between elementary sources is important when studying the structure of acoustic fields corresponding to harmonic (fully coherent) or random (partially coherent) fields. The latter case is the general situation encountered in industrial applications. However, many techniques, such as the holography reconstruction, the identification of models, and active control, are based on the relationships defined for coherent fields. Their adaptation to general fields requires a model for these random fields. Filippi and Mazzoni5 used the cross spectrum between two pressure points to describe these fields and gave a decomposition method in elementary fields.

The purpose of this section is to show how to express an energy field produced by partially coherent sources using a source model composed of elementary sources (monopoles and dipoles). The definition of a coherence matrix of sources in Sec. I is a solution for the representation of a partially coherent field and allows all the energetic quantities to be expressed and calculated. Conditions are given to correctly construct the coherence matrix of sources. In Sec. II, the relationships between partially coherent fields and their decomposed-independent fields are established by the analysis of the principal components of the coherence matrix of sources. It is shown that a partially coherent field can be represented as the superposition of independent fields. Section III presents how to characterize the structure of a partially coherent field using two measurable indicators which are expressed in terms of the coherence matrix of sources: the coherence function between acoustic pressure and particle velocity and the curl of active intensity. Finally we will show that the field matrix defined in this paper can be used to reveal the structure of an acoustic field from all the relationships between the energetic quantities.

I. REPRESENTATIONS OF ENERGY FIELD FROM A SOURCE MODEL

Energetic quantities, such as the active and reactive intensities and the potential and kinetic energy densities, are very useful for studying energy flow and source location. The purpose of this section is to show how to express energy fields of partially coherent sources. First, the fundamental relationships for energetic quantities in a coherent field are reviewed. Then, the expressions for a partially coherent sound field will be developed using a coherence matrix of sources.

A. Fundamental relationships of coherent fields

The time-averaged energetic quantities of an acoustic field can be expressed using the complex notation of the pressure $p$ and the particle velocity $u$. Thus, the active intensity $I$ and the reactive intensity $J$ can be written in the form of a complex intensity $II^*$ as follows:

$$II = I + jJ = \frac{1}{i}p u^*, \quad (1)$$
and the potential and kinetic energy densities \( V \) and \( T \) are expressed, respectively, as

\[
V = \frac{|p|^2}{4 \rho c^2} \quad (2a)
\]

and

\[
T = \frac{\rho}{4} \mathbf{u} \cdot \mathbf{u}. \quad (2b)
\]

where \( \rho \) is the density of the medium and \( c \) is the speed of sound. The relationships between these quantities have been first given by Smith et al.\(^8\)

\[
\nabla \cdot \mathbf{I} = 0, \quad (3a)
\]

\[
\nabla \cdot \mathbf{J} = 2k \varepsilon (V - T), \quad (3b)
\]

\[
\nabla \times \mathbf{I} = -\frac{j \rho c k}{2} \mathbf{u} \times \mathbf{u}^* . \quad (3c)
\]

\[
\nabla \times \mathbf{J} = 0, \quad (3d)
\]

where \( k = \omega / c \) is the acoustic wave number and \( \omega \) is the angular frequency. The spatial distribution of these different quantities has been studied in many configurations.\(^9 \text{–} 13\) The relationship (3d) is a consequence of the fact that the reactive intensity is proportional to the gradient of the potential energy density.\(^10, 14\)

\[
\mathbf{J} = -\frac{c}{k} \nabla V . \quad (4)
\]

The vortex formation in the strong interference region is a particularity of an active intensity field and has received much attention.\(^9, 11, 13, 15 \text{–} 18\) This characteristic of the polarization of the acoustic field bears relation to the curl of the active intensity, which is expressed in terms of the angular momentum density of the fluid particles [see Eq. (3c)].\(^17\) Another expression using only energetic quantities has been derived\(^9, 12\) for coherent fields:

\[
\nabla \times \mathbf{I} = \Omega = \frac{k}{c} \mathbf{I} \times \mathbf{J} . \quad (5)
\]

In most cases only the coherent fields were taken into consideration. Jacobsen\(^19\) shows that the degree of coherence of an acoustic field can be described by the function of coherence between acoustic pressure and particle velocity \( \gamma_{wp}^2 \). This indicator can be expressed by energetic quantities as follows:

\[
\gamma_{wp}^2 = \frac{|G_{wp}|^2}{G_{pp}G_{uu}} = \frac{\mathbf{I} \cdot \mathbf{J}^*}{4c^2 V T}, \quad (6)
\]

where the energetic quantities can be written in the form of power-spectral densities (see Sec. II C). In the case of a coherent field, \( \gamma_{wp}^2 = 1 \), and Eq. (6) provides an additional relationship between the energetic quantities, which allows, for example, to omit kinetic energy \( T \) and to describe the field with only \( \mathbf{I} \) and \( V \) in Eqs. (3). However, in the noncoherent field, there are not obvious solutions to Eqs. (3), (5), and (6). It is necessary to validate these relationships. For this purpose, a model of partially coherent fields generated by a set of elementary sources is used here. In practice, stationary processes are often assumed in the energy field studies. The instantaneous intensity has sometimes been used for understanding the phenomena in acoustic fields,\(^17, 20\) especially for the interpretation of the exchange of energy during one time period in a vortex region\(^3\) or in interference fields.\(^7\) However, for stationary fields, the instantaneous complex intensity does not give supplementary information in comparison with its time-averaged value. Therefore, the time-averaged quantities are only considered in this article.

### B. Representation of partially coherent fields

A field created by a set of harmonic sources is everywhere coherent and the phase between two points in the field is perfectly defined. On the contrary, uncorrelated broadband sources produce partially diffuse acoustic fields. Two field points are not, in general, fully coherent. Hence the phase between two field points can be expressed only for the coherent part of the fields. It is not possible to represent \textit{a priori} the partially coherent fields by the superposition of pressures due to each elementary source. The amplitudes of two elementary sources \( A_i \) and \( A_j \) are stationary random functions whose interdependence relationships can be expressed by a cross-spectral density function:

\[
G_{ij}(\omega) = \lim_{T \rightarrow \infty} \frac{2}{T} \mathbb{E} \{ A_i^*(\omega) A_j(\omega, T) \} , \quad (7)
\]

where \( A_i(\omega, T) \) is the finite Fourier transform of \( A_i(t) \) measured over the finite time interval \( T \).

\[
A_i(\omega, T) = \int_{t_1}^{t_1+T} A_i(t) e^{-j \omega t} dt . \quad (8)
\]

The coherence function between source amplitudes can then be given as

\[
\gamma_{ij}^2(\omega) = \frac{|G_{ij}(\omega)|^2}{G_{ii}(\omega) G_{jj}(\omega)} , \quad (9)
\]

where \( 0 \leq \gamma_{ij}^2(\omega) \leq 1 \) and \( G_{ij}(\omega) \) and \( G_{ij}(\omega) \) are, respectively, auto- and cross-spectral density functions. This coherence function can be used to represent the energetic quantities of broadband acoustic fields using the complex notation. The energetic quantities (intensities and energy densities) in the field generated by a set of \( M \) sources are then expressed in the following forms:

\[
\mathbf{I} = \mathbf{I} + j \mathbf{J} = \frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} \sqrt{\gamma_{ij}^2} \rho \mathbf{u}_i \mathbf{u}_j^* , \quad (10a)
\]

\[
V = \frac{1}{4 \rho c^2} \sum_{i=1}^{M} \sum_{j=1}^{M} \sqrt{\gamma_{ij}^2} \rho \mathbf{p}_i \mathbf{p}_j^* , \quad (10b)
\]

\[
T = \frac{\rho}{4} \sum_{i=1}^{M} \sum_{j=1}^{M} \sqrt{\gamma_{ij}^2} \mathbf{u}_i \mathbf{u}_j^* . \quad (10c)
\]

The relationships between elementary sources are written in terms of a coherence matrix \([ \gamma_{ij}^2 ]\), whose elements \( \gamma_{ij}^2 \) were given by Eq. (9). The matrix \( \gamma_{ij}^2 \) represents the coherence between sources. As examples, consider two typical cases. In
the coherence case, all elements $\gamma_{ij}^2$ of the coherence matrix take the value 1. Only in this case, the complex pressure and particle velocity for the whole field can be defined by $p = \sum_{i=1}^{M} p_i$ and $u = \sum_{i=1}^{M} u_i$. In the case of independent sources, the coherence matrix becomes the identity matrix and the cross-terms are zero, $[\Gamma]^2 = \text{diag}(1,1,\ldots,1)$. The complex intensity is the vectorial sum of that generated by each source: $I = \sum_{i=1}^{M} p_i u_i^* = \sum_{i=1}^{M} (I_i + j J_i)$.

II. COHERENCE MATRIX OF SOURCES

In the previous section it was shown that the coherence matrix of sources is a key to investigating a partially coherent field. However, not any matrix can be used to describe the coherence between sources. The symmetry of $[\Gamma]^2$ is not a sufficient condition for defining the matrix. For example, if two sources have coherent relationships with the third source, the coherence between these two sources depends on these relationships. Therefore, seek the supplementary conditions to correctly construct the matrix $[\Gamma]^2$. The decomposition of this matrix using the principal component analysis allows the necessary numbers of the independent field to be obtained. This analysis finally leads to establishing the relationships between the partially coherent field and the reduced-number-independent fields.

A. Definition of the coherence matrix of sources

The amplitudes of sources $A_i(\omega, T)$ [Eq. (8)] can also be written as

$$A_i(\omega, T) = B_i(\omega) a_i(\omega, T),$$  \hfill (11)

where $B_i(\omega)$ is a complex coefficient and $a_i(\omega, T)$ is the finite Fourier transform of a stationary random signal with the auto-power-spectral density

$$\frac{2}{T} \mathbb{E}[|a_i(\omega, T)|^2] = 1. $$  \hfill (12)

The random signals $a_i(\omega, T)$ are partially coherent among them. Substituting Eqs. (11) and (12) into Eq. (9) yields

$$\gamma_{ij}^2(\omega) = \frac{2}{T} \mathbb{E}[a_i^*(\omega, T) a_j(\omega, T)]^2.$$  \hfill (13)

To obtain all elements of the matrix $[\Gamma]^2$, $a_j(\omega, T)$ can be expressed as the linear combinations of $M$ independent stationary random signals $s_m(\omega, T)$:

$$a_j(\omega, T) = \sum_{m=1}^{M} \xi_{jm}(\omega) s_m(\omega, T). \hfill (14)$$

According to Eq. (11), each real source is the superposition of a number of independent sources. Uncorrelated random signals $s_m(\omega, T)$ have the following property:

$$\frac{2}{T} \mathbb{E}[s_m^*(\omega, T) s_n(\omega, T)] = \delta_{mn} = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m, \end{cases} \hfill (15)$$

and the coefficients $\xi_{jm}(\omega)$ are real numbers such that $0 \leq \xi_{jm}(\omega) \leq 1$. From Eqs. (12), (14), and (15), the coefficients $\xi_{jm}(\omega)$ should satisfy the following condition:

$$\sum_{m=1}^{M} \xi_{jm}^2(\omega) = 1. $$  \hfill (16)

A new expression of mutual coherence of sources is obtained by substituting Eq. (14) into Eq. (13)

$$\gamma_{ij}^2(\omega) = \left( \sum_{m=1}^{M} \sum_{n=1}^{M} \xi_{im}(\omega) \xi_{jn}(\omega) \delta_{mn} \right)^2 = \left( \sum_{m=1}^{M} \xi_{im}(\omega) \xi_{jn}(\omega) \right)^2.$$  \hfill (17)

The mutual coherence matrix $[\Gamma]^2$ is hence expressed by the matrix $[\xi]$ as follows:

$$[\Gamma]^2 = ([\xi][\xi]^T),$$  \hfill (18)

where $[\cdot]^T$ designates the transpose of the matrix. The matrix $[\Gamma] = [\xi][\xi]^T$ whose coefficients are $\gamma_{ij}^2$ is equally well used as the coherence matrix of sources. According to Dodds and Robson, the matrix $[\xi]$ that will correctly construct the matrix $[\Gamma]^2$ is a lower triangular matrix such as

$$\begin{bmatrix} a_1(\omega, T) \\ a_2(\omega, T) \\ \vdots \\ a_M(\omega, T) \end{bmatrix} = \begin{bmatrix} \xi_{11} & 0 & \cdots & 0 \\ \xi_{21} & \xi_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{M1} & \xi_{M2} & \cdots & \xi_{MM} \end{bmatrix} \times \begin{bmatrix} s_1(\omega, T) \\ s_2(\omega, T) \\ \vdots \\ s_M(\omega, T) \end{bmatrix}, \hfill (19)$$

and the only constraint of this definition is the normalization on each line according to Eq. (16) (which results in $\xi_{11} = 1$ in all cases).

As examples, consider two typical cases. In the case of $M$ independent sources, it is evident that $a_i(\omega, T) = s_i(\omega, T)$ and that the matrix $[\xi]$ is an identity matrix $[\xi]=\text{diag}(1,1,\ldots,1)$. In the case of $M$ coherent sources, each random signal $a_i(\omega, T)$ should be represented by the same linear combination $\sum_{m} \xi_{im}(\omega) s_m(\omega, T)$, which results in a matrix having only one nonzero column by using the definition of the triangular matrix (19). In Ref. 22 it was shown how to construct $[\xi]$ for a particular case where the coherence functions among sources are all identical: $\gamma_{ij}^2(\omega) = g^2$ for $j \neq i$, $i,j = 1,2,3,\ldots,M$.

B. Analysis of the coherence matrix

The coherence matrix of sources can be determined by the coefficient matrix $[\xi]$. A method is shown here to obtain the independent fields from the coherence matrix of sources using the singular value decomposition (SVD).

1. Decomposition into independent fields

The rank of $[\xi]$ corresponds to the number of independent columns, therefore, to the number of independent primary sources $s_m(\omega, T)$ necessary to produce the signals
A_i(\omega, T). In the coherent case each random signal a_i(\omega, T) should be represented by the same linear combination \sum_n \xi'_n(\omega) s_m(\omega, T). Thus the matrix [\xi]\xi^T has only one non-zero column: its rank is equal to 1, therefore only one primary source s_1(\omega, T) is needed. On the contrary, in the case of non-coherence, [\xi]\xi^T is an identity matrix, [\xi]\xi^T = \text{diag}(1,1,\ldots,1), and its rank is equal to M. In the case of partially coherent sources, one encounters, in practice, many of these intermediate situations where one or several columns are linear combinations of the others. It is thus possible, in principle, to reduce the number of independent sources by using the principal component analysis (PCA).\textsuperscript{23,24} Calculating the singular value decomposition (SVD) of the matrix [\Gamma]\xi^T results in the following factorization:

$$[\Gamma]_{M \times M} = [H]_{M \times M}[\Sigma^2]_{M \times M}[H]^T_{M \times M}. \quad (20)$$

The diagonal matrix [\Sigma^2] contains the eigenvalues of [\Gamma] arranged in decreasing order,

$$[\Sigma^2]_{M \times M} = \text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_r^2, 0, \ldots, 0). \quad (21)$$

If the matrix [\Gamma] has r nonzero eigenvalues, its rank is r. It can be shown that for a Hermitian matrix, these eigenvalues are all real and positive. The columns of the orthonormal matrix [H] correspond to the complex eigenvectors, in which only the first r columns are significant. Therefore, if r < M, the sizes of the matrices [\Sigma^2] and [H] can be reduced without loss of any information such that

$$[\Gamma]_{M \times M} = [H']^T_{M \times r}[\Sigma^2']_{r \times r}[H']_{r \times M}, \quad (22)$$

where [\Sigma^2']_{r \times r} = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r). From the reduced matrices [\Sigma^2']_{r \times r} and [H']_{M \times r}, a reduced coefficient matrix [\xi'] can be constructed:

$$[\xi']_{M \times r} = [H']_{M \times r}[\Sigma^2']_{r \times r}^{1/2}. \quad (23)$$

The replacement of [\xi] with [\xi'] means, according to Eq. (19), that the random signals a_i(\omega, T) representing the M partially coherent sources can be reconstituted from only r independent source signals s_m(\omega, T), (m = 1,2,3,\ldots,r).

### 2. Example

Figure 1 shows a source model synthesized by four monopoles (S_1, S_2, S_3, S_5) and one dipole (S_4) directing to the view plane. These five point sources have the same amplitude and are in phase. To generate a coherence matrix of a source in the general case, consider a matrix [\xi]_{3 \times 5} that satisfies Eqs. (16) and (19), but one column of which is a linear combination of the others. The coherence matrix of sources [\Gamma^2] can be calculated using Eq. (18). The cross-term values of [\Gamma^2] are represented in Fig. 2. The diagonal matrix of eigenvalues is obtained by calculating the SVD of the matrix [\xi][\xi]^T.

$$[\Sigma^2] = \text{diag}(3.1399, 1.0251, 0.8075, 0.0275, 0.0000), \quad (24)$$

with \epsilon = 3 \times 10^{-16} which can be neglected in comparison with the sum \sum_{j=1}^{M} \sigma_j^2 = M = 5. Since two columns in [\xi] are not independent, the rank of [\xi][\xi]^T is equal to 4 (inferior to M = 5). The reduced matrix [\xi']_{3 \times 4} is obtained using Eq. (23). Then [\xi'] is used to calculate the coherence matrix of sources: [\Gamma^2'] = ([\xi'][\xi']^T)^2. It can be verified that it is identical to the initial matrix. The triangular matrix [\xi] is replaced by a reduced matrix [\xi'] with elements varying between -1 and 1.

To generalize the previous discussion, consider the first three eigenvalues of the matrix (24) which already take 99.45% of the total energy. The matrix of coherence deduced from the three components is calculated from a normalized matrix [\xi'']_{3 \times 3} whose elements are obtained by \xi''_{im} = \xi'_{im} / \sqrt{\sum_{j=1}^{M} \xi'_{ij}^2}^{-1/2} to satisfy the condition (16). It results in an approximate expression of the initial coherence matrix

$$[\Gamma'^2] = ([\xi'']^T[\xi'']^T)^2. \quad (25)$$

If only the first two eigenvalues of the diagonal matrix (24) are taken into account, another approximation of the coherence matrix can be obtained from the normalized matrix [\xi'']_{3 \times 3}:

$$[\Gamma'^2] = ([\xi'']^T[\xi'']^T)^2. \quad (26)$$

The cross terms of approximate coherence matrices given by Eqs. (25) and (26) are compared to those of the initial matrix in Fig. 2. The matrix [\Gamma'^2] gives a good approximation to [\Gamma^2], but [\Gamma'^2] is very different from the initial matrix because a component of the independent field which represents...
approximately 16.7% of sound field energy has not been considered.

The active intensity and potential energy density fields created by the synthetic source in Fig. 1 at 1000 Hz were calculated using Eq. (10). The initial coherence matrix of sources shown in Fig. 2 and its approximate expressions Eqs. (25) and (26) were used, respectively, in the calculations. The results are shown in Fig. 3(a)–(c). Graphically, the potential energy density is represented by grey-scale plots. The three-dimensional active intensity vectors are presented with arrows which indicate the three components. The in-plane components are represented in amplitude by the lengths of arrows and in direction by the orientations of arrows. The out-of-plane component is represented in amplitude by the thicknesses of arrows and in direction by a filled arrow representing $I_z$ pointing out of the view plane and an empty arrow pointing into the view plane. Comparing the results in Fig. 3(a) and (b), it is noted that Eq. (25) $[T^{12}]$ gives a very good approximation to the initial coherence matrix of sources $[T^{22}]$ for the calculation of the energetic quantities $I$ and $V$. However, Fig. 3(c) shows the differences from the results in Fig. 3(a) and (b) because the third independent component cannot be negligible. Nevertheless, the structure of the field is not fundamentally changed and the deviations have approximately the same order as the measurement errors in industrial situations. The use of only three independent components instead of four has practical interests, for example, in the experimental realization of a synthetic source using a $\xi$-matrix where only three noise generators will be necessarily needed (see Fig. 4). A real field can often be recreated from a synthetic field composed of a reduced number (often less than 5) of independent principal components.

C. Partially coherent fields and independent fields

Equation (10a) representing active and reactive intensities of partially coherent fields can also be expressed in the matrix form:

![Diagram](https://via.placeholder.com/150)

FIG. 4. Realization of a synthetic source from coefficients $\xi_{ij}$ obtained by principal component analysis of the matrix of coherence of sources. $LS_i$: excitators or loudspeakers, $G_j$: independent noise generators.
the column matrices
\[
\begin{bmatrix}
p_m \\ \vdots \\ p_M
\end{bmatrix}
\]
and the velocity
\[
\begin{bmatrix}
u_m \\ \vdots \\ \nu_M
\end{bmatrix}
\]
are represented as the superposition of independent fields. Thus these independent fields are termed ‘virtual fields.’

III. CHARACTERIZATION OF PARTIALLY COHERENT ACOUSTIC FIELDS

Generally, for an acoustic field created by multiple sources, it is not possible to directly estimate the mutual coherence between sources from measurements. However, the information about the degree of coherence of the acoustic field is important to evaluate measurement conditions, statistical errors, or the validity of the use of the algorithms derived from harmonic signals. The coherence function between pressure and particle velocity \( \gamma_{pu}^2 \) and the curl of acoustic intensity \( \nabla \times \mathbf{I} \) are used here to characterize the structure of acoustic fields. It is shown later that the complete information about the local structure of acoustic fields is provided by the definition of a field matrix.

A. Coherence function of sound fields

The superposition of independent fields leads to a loss of coherence between the acoustic pressure and the particle velocity. The characteristic of the acoustic field is represented by the coherence function \( \gamma_{wp}^2 \) defined by Eq. (6), which is expressed in terms of measured (or calculated) energetic quantities in random acoustic fields. By using Eq. (10) associated with the coherence matrix of sources or Eq. (30) for \( r \) independent fields, the coherence function of an acoustic field takes the following form:

\[
\gamma_{wp}^2(\mathbf{r}, \omega) = \frac{\sum_{i=1}^{M} \sum_{j=1}^{M} \sqrt{\gamma_{ij}} \mathbf{p}_i \mathbf{u}_j^*}{\sum_{i=1}^{M} \mathbf{p}_i \mathbf{u}_i^*}
\]

where \( \gamma_{ij} \) is the coherence function derived from harmonic signals. The coherence function between pressure and particle velocity \( \gamma_{pu}^2 \) and the curl of acoustic intensity \( \nabla \times \mathbf{I} \) are used here to characterize the structure of acoustic fields. It is shown later that the complete information about the local structure of acoustic fields is provided by the definition of a field matrix.

If we consider the source model in Fig. 1 without the source \( S_5 \), Fig. 5 demonstrates that the distribution of \( \gamma_{pu}^2 \) on...
the view plane (at 1000 Hz) is very sensitive to the values of the coherence matrix of sources [$\gamma_{ij}^2$]. The experimental determination of this indicator from Eq. (6) will therefore provide the information about the nature of the radiating acoustic field, as well as about the source which is at the origin, as an example, a real source with a set of independent excitations. In practice, the coherence function between the pressure and a component of the particle velocity is often used to control the validity of the measurements of the corresponding component of the intensity vector. The coherence function between the pressure and $x$-component velocity, for example, is written as $\gamma_{u_p, \alpha}^2 = \langle T_{xx}^2 + 2T_{xy}T_{xy} + T_{yy}^2 \rangle / T_{xx}^2$, with $T_{xx} = \rho [u_1]^2 / 4$. The coherence function of the field [Eq. (16)] can be, therefore, written as

$$\gamma_{wp}^2 = \frac{\gamma_{u_p, \alpha}^2 T_{xx} + \gamma_{u_p, \alpha}^2 T_{yy} + \gamma_{u_p, \alpha}^2 T_{zz}}{T_{xx} + T_{yy} + T_{zz}}. \quad (32)$$

In Fig. 7, the coherence function was determined experimentally in a horizontal line on the view plane, 4 cm from a baffle on which two loudspeakers were mounted [see Fig. 6(a)]. The experimental system was set up in an anechoic chamber. The components of the active intensity in the view plane parallel to the baffle were measured using a two-dimensional intensity probe consisted of four quarter-inch microphones [Fig. 6(b)] positioned by an automatic robot system. The coherence function calculated by

$$\gamma_{wp, \text{plan}}^2 = \frac{\gamma_{u_p, \alpha}^2 T_{xx} + \gamma_{u_p, \alpha}^2 T_{yy}}{T_{xx} + T_{yy}} \quad (33)$$

is an incomplete form of Eq. (32), but it remains valid ($0 \leq \gamma_{wp, \text{plan}}^2 \leq 1$) to characterize the measured data, for example, used in the acoustic holography processing. The energetic quantities are calculated using the well-known finite-difference approximation, for example, $I_z = \text{Im} \{G_{23}^2(\omega) \} / \rho_0 \Delta r$ and $J_z = \{ G_{11}(\omega) - G_{22}(\omega) \} / 2 \rho_0 \omega \Delta r$, where $\Delta r$ is the distance between the two microphones shown in Fig. 6(b), $G_{11}$ and $G_{22}$ are the auto-power-spectral densities of signals measured by microphone 1 and microphone 2 respectively, and $G_{21}$ is the cross-power-spectral density between the signals of microphones 1 and 2. A multichannel signal processing system (LMS Cada-X) was used for data acquisition. A mixer was used to obtain the coherence $\gamma_{ij}^2$ between two white noise signals built in (LMS Cada-X) by adjusting the amplitudes of the signals. Then the two mixed signals were used to drive the two loudspeakers. The energetic quantities used in Eq. (33) were determined with a frequency resolution $\Delta f$ of 3.9 Hz. Figure 7 illustrates the coherence of field $\gamma_{wp}^2$ measured for different values of the coherence $\gamma_{ij}^2$. The results presented at 281 Hz and at 1496 Hz show, when $\gamma_{ij}^2 < 1$, a sensitive coherence loss in the central area where the two loudspeakers were mounted.

**B. Indicator of the polarization of fields**

The polarization vector for a coherent field was defined by Uosukainen$^{17}$ as

$$a = - j \frac{\mathbf{u} \times \mathbf{u}^*}{\mathbf{u} \cdot \mathbf{u}^*}, \quad (34)$$

where $0 \leq |\mathbf{a}| \leq 1$. The vector $\mathbf{a}$ is zero when the particle velocity is polarized linearly, i.e., the orientation of its instantaneous value is always the same during one time period (vectors of the active and reactive intensities are parallel with each other). In general, the velocity field has an elliptic polarization, that is, the orientation of its instantaneous particle velocity changes during one time period and traces an el-

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**FIG. 6.** (a) Plane of baffle and positions of loudspeakers. (b) Two-dimensional acoustic intensity probe with four microphones used in the measurements.

**FIG. 7.** Coherence function of fields measured in the view plane parallel to the baffle of Fig. 6 for values $\gamma_{ij}^2$ indicated; (a) $f = 281$ Hz, (b) $f = 1496$ Hz (frequency resolution = 3.9 Hz).
ellips. The instantaneous velocity has often been decomposed into two parts: one is in phase and another is in quadrature with the pressure. The directions of the two components correspond to those of active and reactive intensities, respectively. Thus the magnitude of the polarization vector becomes maximum (equal to 1) when the two components of the particle velocity are the same magnitude but perpendicular (the ellipse becomes a circle). According to Eq. (3c), the polarization vector is the curl of acoustic intensity normalized by kinetic energy

$$a = \frac{\nabla \times I}{2\kappa c T}.$$  \hspace{1cm} (35)

It can be seen that a field is hence polarized when the curl of intensity is not zero. The curl, i.e., the polarization, depends strongly on interference between sources. Thus, the field of a monopole possesses a zero curl. Several coherent monopoles will produce an interference field in which the curl vector characterizes the polarization of the field. On the contrary, if the monopole sources are independent, the zero curl indicates a nonpolarized field. In an intermediate situation, the curl measures the polarization of the coherent part. Thus, by calculating the curl of Eq. (10a) with the consideration of the relationships $\nabla p_i = -j\rho c \mathbf{u}_i$ and $\nabla \times \mathbf{u}_i = 0$ (ideal nonviscous fluid), one obtains

$$\nabla \times I = -\frac{j\rho c}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} \sqrt{\gamma_{ij}} \mathbf{u}_i \times \mathbf{u}_j^*.$$  \hspace{1cm} (36)

Using the symmetric relationships between $\mathbf{u}_i \times \mathbf{u}_j^*$ and $\mathbf{u}_j \times \mathbf{u}_i^*$, Eq. (36) becomes

$$\nabla \times I = \nabla \times I = \frac{\kappa}{2} \text{Im} \left[ \sum_{i=1}^{M} \sum_{j=1}^{M} \sqrt{\gamma_{ij}} \mathbf{u}_i \times \mathbf{u}_j^* \right].$$  \hspace{1cm} (37)

Multiplying the numerator and the denominator in Eq. (37) by $p_i^* p_j$ yields

$$\frac{\kappa}{2} \text{Im} \left[ \mathbf{u}_i \times \mathbf{u}_j^* \right] = \frac{\kappa}{2} \text{Im} \left[ \frac{\Pi_i^p \times \Pi_j^p}{V_{ij}} \right],$$  \hspace{1cm} (38)

with $\Pi_i^p = p_i \mathbf{u}_i^p/2$ and $V_{ij} = p_i^* p_j / 4 \rho c^2$. Using $(\Pi_i^p \times \Pi_j^p) + (\Pi_j^p \times \Pi_i^p) = 2j(\mathbf{I}_i \times \mathbf{I}_j) + 2j(\mathbf{I}_j \times \mathbf{I}_i)$, Eq. (38) can also be expressed in the form of the vector product of the active intensity and the reactive intensity of each individual source:

$$\nabla \times I = \frac{\kappa}{2} \sum_{m=1}^{r} \sum_{n=1}^{r} \frac{\mathbf{I}_m \times \mathbf{I}_n}{V_{mn}}.$$  \hspace{1cm} (39)

Substituting values $\gamma_{ij} = 1$ into Eq. (37), we get the expression of the curl of a coherent field ($\gamma_{\alpha\beta} = 1$)

$$\nabla \times I \big|_{\gamma_{ij} = 1} = \frac{\kappa}{2} \text{Im} \left[ \sum_{i=1}^{M} \sum_{j=1}^{M} \mathbf{u}_i \times \mathbf{u}_j^* \right],$$  \hspace{1cm} (40)

where $\mathbf{I}_m$ and $\mathbf{I}_n$ are defined by Eqs. (10a) and (10b) when $\gamma_{ij} = 1$. In these conditions, according to Eq. (5) the polarization vector of the coherent field can be expressed as follows:

$$a \big|_{\gamma_{ij} = 1} = \frac{\mathbf{I} \times \mathbf{J}}{2\kappa c^2 V T}.$$  \hspace{1cm} (41)

In order that $|a| = 1$, the active and reactive intensity vectors should be perpendicular to each other and $|\mathbf{I}| = |\mathbf{J}|$. Equation (39) demonstrates that the curl of a partially coherent field cannot be calculated from vector product of $\mathbf{I}$ and $\mathbf{J}$ and that Eq. (5) is valid only for a coherent field. This is illustrated obviously in Fig. 8 where the curl of the field produced by the source model in Fig. 1 without the source $S_5$ is calculated by Eqs. (39) and (5) respectively in the cases where $\gamma_{ij} = 0$, 0.25 and 1. It is noted that Eqs. (5) and (39) give the same results when $\gamma_{ij} = 1$. There are, however, differences between the results calculated by Eq. (5) and by Eq. (39) when $\gamma_{ij} \neq 1$. When $\gamma_{ij} = 0$, it is obvious that Eq. (5) gives wrong results for the calculation of the curl of the intensity because in this case no interference between the sources exists and the polarization of the field is only due to the dipole source. The invalid use of Eq. (5) in noncoherent acoustic fields makes the nonexistent components appear, especially outside the representation plane. To explain this phenomenon, a field constituted by the superposition of two nonpolarized independent coherent fields ($\nabla \times \mathbf{I}_i = 0$, therefore $\mathbf{I}_i \times \mathbf{J}_i = 0$, $i = 1, 2$) is considered. The resultant active and reactive intensities in this field are written, respectively, as $\mathbf{I}_i = \mathbf{I}_1 + \mathbf{I}_2$ and $\mathbf{J}_i = \mathbf{J}_1 + \mathbf{J}_2 = a \mathbf{I}_1 + b \mathbf{I}_2$, where $a$ and $b$ are two scalar factors. In the general case when $a \neq b$, these resultant vectors are not parallel and $\mathbf{I} \times \mathbf{J}$ is not zero while the curl of the resultant field is always equal to 0.

By the use of the principal component decomposition to a partially coherent field according to Eq. (29), the curl of the resultant vector corresponds to the sum of the curl of the elementary vectors

$$\nabla \times \mathbf{I} = \sum_{m=1}^{r} \nabla \times \mathbf{I}_m = \sum_{m=1}^{r} \frac{\mathbf{I}_m \times \mathbf{J}_m}{V_m}.$$  \hspace{1cm} (42)

Similarly, the polarization vector of the field can be determined from the polarization vector and the kinetic energy density of each elementary field

$$a = \frac{\sum_{m=1}^{r} T_m \mathbf{a}_m}{T}.$$  \hspace{1cm} (43)

With regard to the experimental determination of the curl for partially coherent fields, Eq. (3c) is adapted to random signals by

$$\nabla \times \mathbf{I} = \rho c \lim_{T \to \infty} \frac{2}{T} \text{Im} \left[ \mathbf{E}[\mathbf{u}(\omega, T) \times \mathbf{u}^*(\omega, T)] \right],$$  \hspace{1cm} (44)

where $\mathbf{u}(\omega, T)$ is the finite Fourier transform defined by Eq. (8). Equation (44) can also be given as

$$\nabla \times \mathbf{I} = 4\epsilon c \lim_{T \to \infty} \frac{2}{T} \text{Im} \left[ \mathbf{E}[\mathbf{u}(\omega, T) \times \mathbf{u}^*(\omega, T)] \right],$$  \hspace{1cm} (45)

where $T_{ij} = (\rho/2) \lim_{T \to \infty} \frac{2}{T} \text{Im} \left[ \mathbf{E}[\mathbf{u}(\omega, T) \times \mathbf{u}^*(\omega, T)] \right]$, with $i, j \in \{x, y, z\} \ (i \neq j)$. Using the same configuration mentioned above to determine the coherence function $\gamma_{\alpha\beta}$plane, the normal component of the curl is measured and calculated by the following expression derived from Eq. (45):
and the six cross-power-spectral densities shown in Fig. 6 using Eq. (46).

Expressing the x component and y component of the pressure gradient in terms of the cross-power spectra $G_{ij}(\omega); i, j = 1, 2, 3, 4; i \neq j$ between the microphones of the two-dimensional sound intensity probe [see Fig. 6(b)] by the use of the finite-difference approximation, Eq. (46) becomes

$$\left( \nabla \times \mathbf{I} \right)_x = \frac{2}{\rho c k} \lim_{r \to \infty} \frac{2 \, \text{Im} \left\{ \mathbf{E} \left[ \frac{\partial p}{\partial x} \right]^* \right\}}{r^2}.$$  \hspace{1cm} (46)

where $\Delta r_1 = \Delta r/\sqrt{2}$ is the distance between the two microphones shown in Fig. 6(b). In the case of a coherent field, the normal component of the curl of the intensity can be derived using Eq. (5):

$$\Omega_z = \frac{k}{c} \sum_{i=1}^{4} \frac{\mathbf{J}_i \cdot \mathbf{J}_j}{\mathbf{V}},$$  \hspace{1cm} (47)

where $\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3, \mathbf{J}_4$, and $\mathbf{V}$ are determined by standard formulations using the finite-difference approximation. In order to validate Eqs. (47) and (48), the curl of the acoustic intensity was measured using the experimental set-up shown in Fig. 6. The experimental procedure was similar to that described in Sec. III A. The intensity probe was used to measure the four auto-power-spectral densities $G_{ii}(i=1, 2, 3, 4)$ and the six cross-power-spectral densities $G_{ij}(i > j, i, j = 1, 2, 3, 4)$ which were then used to calculate the curl of the intensity using, respectively, Eqs. (47) and (48). Figure 9 shows $\left( \nabla \times \mathbf{I} \right)_z$ and $\Omega_z$ for the cases where $\gamma_{21}^2 = 0$ and $\gamma_{21}^2 = 0.5$. It is noted that there are differences between the results of Eqs. (47) and (48). The differences are larger when $\gamma_{21}^2 = 0$. The significant differences are found in the central area where the values of $\gamma_{21}^2$ are small and the two loudspeakers were mounted.

**C. The field matrix**

Locally, a coherent field can be completely described by quantities $p, u_x, u_y$, and $u_z$. Their relationships lead to an energetic representation of the field which can be generalized in the form of the field matrix:

$$\mathbf{F} = \frac{3}{2} \mathbf{\Theta} \mathbf{\Theta}^T,$$  \hspace{1cm} (49)

where $\mathbf{\Theta} = [p, \sqrt{p}c, \sqrt{p}c u_x, \sqrt{c}u_y, \sqrt{c}u_z]^T$ is the column vector of acoustic quantities normalized in order to have the same dimension. The field matrix can be expressed by the energetic quantities as

$$\mathbf{F} = \begin{bmatrix} 2cV & \Pi_x & \Pi_y & \Pi_z \\ \Pi_x^* & 2cT_{xx} & 2cT_{xy} & 2cT_{xz} \\ \Pi_y^* & 2cT_{xy} & 2cT_{yy} & 2cT_{yz} \\ \Pi_z^* & 2cT_{xz} & 2cT_{yz} & 2cT_{zz} \end{bmatrix}.$$  \hspace{1cm} (50)
It has a Hermitian symmetry ($F_{ij} = F_{ji}^*$) and the sum of its diagonal elements corresponds to the total energy of the field: $2c(V + T)$. For a partially coherent sound field, the vector $[\Theta]$ cannot be defined and the matrix $[F]$ must be characterized by the cross-power spectra $\lim_{T \to \infty} (2/T) \mathcal{E}[\theta^A(\omega,T) \theta^B(\omega,T)]$, some of which have already been used in the expressions of the two previous indicators: the cross spectra between pressure and particle velocity for the coherence function of the field $\gamma_{ap}$ and the imaginary parts of cross spectra between components of velocities for the curl. In what follows below, some properties of the field matrix are proven by choosing the coordinates $(x',y',z')$ where the axis $x'$ is in the direction of the vector of the active intensity $I$ and the plane $(x',y')$ corresponds to the plane defined by vectors $I$ and $J$. The field matrix obtained in these particular coordinates is denoted by $[F_0]$, namely, the principal field matrix.

1. Case of a coherent field

Equation (5) is used to calculate the curl of the intensity of a coherent field. In the coordinates defined above, the curl of $I$ is oriented in the direction $z'$ and is expressed by

$$\nabla \times I = 4ck \text{ Im}[T_{x'y'}] = jk(F_{z1} - F_{21})$$

with $2c T_{x'y'} = -\rho u_{x'} u_{y'}^* / 2$. The field matrix of a coherent field is written, therefore, in the following form:

$$[F_0] = \begin{bmatrix}
2cV & I_{x'} + jJ_{x'} & jJ_{y'} & 0 \\
I_{x'} - jJ_{x'} & 2cT_{x'y'} & 2cT_{x'y'} & 0 \\
-jJ_{y'} & 2cT_{x'y'} & 2cT_{y'y'} & 0 \\
0 & 0 & 0 & 0 
\end{bmatrix}. \quad (51)$$

In this configuration, the reactive intensity that forms an angle $\alpha$ with the active intensity has the components $J_{x'}$ and $J_{y'}$ (except for particular cases). An analytic expression of the particle velocity can be given by $u_{x'} = u_{x'R} - j u_{x'T}$, where $u_{x'R}$ and $u_{x'T}$ are, respectively, in phase and in quadrature with the pressure. The active and reactive intensities can be thus written as $I_{x'} = \frac{1}{2}pu_{x'} + J_{x'}$, $I_{y'} = \frac{1}{2}pu_{x'}$. In the coordinates $(x',y',z')$, $I_{x'} = 0$, $u_{x'}$ is in quadrature with $p(u_{x'} = -ju_{y'})$ and hence $J_{y'} = (1/2)p u_{y'}$. Similarly, $2c T_{x'y'} = \rho p u_{x'} u_{y'}^* / 2 = \rho c(u_{x'} u_{y'} - j u_{x'} u_{y'}) / 2$. The curl corresponding to the imaginary part of this last relationship will always be oriented in the direction of the $z'$ axis. In the special case where the coherent field is nonpolarized, the velocity only has a component in the direction of $x'$ and $J_{y'} = 0$, $T_{x'y'} = 0$ and $T_{y'y'} = 0$.

2. Case of a partially coherent field

Since any partially coherent field can be represented by the sum of $r$ independent coherent fields, all elements of the matrix $[F_0]$ will have a nonzero value, except for $F_{11} = F_{13} = 0$, which is the consequence of the choice of the coordinates system: there is no $z'$ component of the resultant intensity vector, however, in general, $T_{z'x'} \neq 0$. If the independent fields are nonpolarized we have noted previously that a component $J_{z'}$ exists but the imaginary parts of $T_{z'x'}$, $T_{z'y'}$, and $T_{z'y'}$ are zero. This will not be true in the general case, where $\text{Im}[T_{z'x'}]$ and $\text{Im}[T_{z'y'}]$ are not zero: the curl of the resultant vector $I$ is no longer oriented in the direction $z'$, as shown in Fig. 10.

3. Use of the rank of the field matrix

A singular value decomposition of the field matrix $[F]$ can also constitute a local indicator of the nature of the acoustic field. In the coherent case, the rank of the field matrix is equal to 1 and the eigenvector is proportional to the vector $[\Theta]$. For a partially coherent field, the rank of the field matrix may be 2, 3, or 4, which can be considered as a local
indicator of the complexity of the acoustic field. When the rank is superior to 2, the loss of coherence between pressure and particle velocity is accompanied by a loss of coherence between the orthogonal components of the particle velocity.

IV. CONCLUSION

Any random acoustic field can be represented by a model consisting of point sources or elementary fields whose relationships are defined by the coherence matrix of sources. The conditions that are required to correctly construct this matrix have been specified. Energetic quantities in a partially coherent field are expressed in terms of the coherence matrix of sources. It is shown that an acoustic field can be expressed as the superposition of independent fields which are coherent themselves but independent of each other. The investigation of the coherence matrix using the principal component analysis allows the number of the independent components to be minimized. The approximate model can be used for designing acoustic experiments, suppressing the weak components without appreciably modifying the structure of the field. To obtain measurable characteristics of the degree of coherence of an acoustic field, the function of coherence between the pressure and the particle velocity can be used. It is completed by the information about the polarization of the field in the form of an indicator determined from the measurement of the curl of acoustic intensity. It is shown that the curl of the intensity of a partially coherent field cannot be expressed in terms of the vector product of the resultant active and reactive intensities and Eq. (5) is valid only for a coherent field. Finally, we have shown that the definition of a field matrix made up of the relationships between the acoustic pressure and the three components of the particle velocity can completely take into account the local structure of an acoustic field and the rank of the field matrix can be used as an indicator of the local complexity of the acoustic field.