A NEW K-SPACE OPTIMAL FILTER FOR
ACOUSTIC HOLOGRAPHY

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ABSTRACT — The existence of noise in evanescent waves associated with backward propagation in acoustic holography is a cause of distortion of sound field reconstructions. An effective way is to apply a K-space filter to data set on the hologram. The use of these parametric windows remains empirical. In the present work, a more systematic approach is adopted by using a least-squares optimisation method combined with a condition whereby the integral of the total acoustic energy density (potential and kinetic) on the source plane is made to converge. This new optimal K-space filter obtained no longer depends on arbitrary parameters but on factors defined by the experimental conditions, such as the signal to noise ratio, the distance from the hologram to the source plane and the frequency of analysis. This window is compared with an existing empirical window.

INTRODUCTION

The technique of acoustic holography makes it possible to reconstruct a three-dimensional acoustic field or the image of acoustic sources from measurements in two dimensions on a surface (the hologram). Spatial fluctuations at the source that are smaller than the wavelength in air, $\lambda = 2\pi c/\omega$, are unable to propagate to the far field. In wavenumber space, they correspond to evanescent waves which decay exponentially with distance from the source. It is these evanescent waves which create the field structure close to the sources and which are needed to reconstruct the source field details smaller than $\lambda/2$ [1, 2, 3]. In the process of reconstructing the source field, the evanescent components are amplified exponentially. These particular procedures introduce instability to the process of backward propagators which is characteristic of the "ill-posed" nature of the inverse problem. This instability appears especially when the measurement or calculational noise contaminates the evanescent waves in the plane of the hologram. The high order evanescent components, which decay more quickly, fall below the noise threshold. To avoid amplifying the noise instead of the signal during the backward process, it is essential to delete the components carrying the greatest distortion. Different authors [2, 4, 5, 6] have suggested a number of filter windows, and that proposed by Veronesi and Maynard [4] is the most often used. However the use of these windows is still highly empirical. To begin with, no criteria exist for choosing
parameters, especially as regards the cut-off wavenumber \( K_c \). In this article, a least squares optimisation method is used to define filter windows, which are then compared with the more empirical approaches proposed elsewhere.

**PROBLEMS ENCOUNTERED IN RECONSTRUCTING THE FIELD**

**Evanescent waves and measurement noise**

In plane geometry, the process of reconstructing the acoustic pressure at the source plane \( p(x, y, z_S) \) using acoustic holography is described by the following equation

\[
p(x, y, z_S) = \mathcal{F}^{-1}\left\{ p_M(K_x, K_y, z_H) G_p^{-1}(K_x, K_y, d) W(K_x, K_y) \right\}. \tag{1}
\]

The operator \( \mathcal{F}^{-1}\{ \} \) represents the inverse Fourier spatial transformation and \( p_M(x, y, z_H) \) corresponds to the spatial Fourier transform of the complex pressure measured on the hologram plane. The pressure propagation operator \( G_p \) is the solution of the equation obtained by taking the spatial Fourier transform on the variables \( x \) and \( y \) of the homogeneous Helmholtz equation

\[
\frac{\partial^2 p(K_x, K_y, z)}{\partial z^2} + (k^2 - K_x^2 - K_y^2) p(K_x, K_y, z) = 0. \tag{2}
\]

The inverse operator of Eq. (1) is therefore defined by

\[
G_p^{-1}(K_x, K_y, d) = \begin{cases} \exp\left[ j d \sqrt{k^2 - K_x^2} \right], & \text{for } K_x^2 \leq k^2 \\ \exp\left[ d \sqrt{K_x^2 - k^2} \right], & \text{for } K_x^2 > k^2, \end{cases} \tag{3}
\]

where \( d = z_H - z_S \) is the distance from the hologram to the source plane and \( K_x^2 = K_x^2 + K_y^2 \). This equation shows that the K-space where the calculation is done is divided into two separate parts by the circle of radius \( k = \omega/c \) (see Figure 1). Inside the circle, each component is propagated as a plane wave, i.e., with phase rotation but without amplitude attenuation. Outside the circle, it is the evanescent components which decay exponentially away from the source surface. But these evanescent components are necessary to obtain resolution much better than a half wavelength. The process of inverse propagation will amplify them exponentially, which makes the operator \( G_p^{-1} \) unstable. In fact, assuming that measurement noise \( p_B(z_H) \) is introduced during acquisition of the hologram and uniformly distributed in the K-space, the integral

\[
\int \int_{-\infty}^{\infty} \left| p_B(z_H) G_p^{-1}(K_x, K_y, d) \right|^2 \frac{dK_x}{2\pi} \frac{dK_y}{2\pi} \tag{4}
\]

does not correspond to a finite potential energy at the source plane. For this reason, Eq. (1) includes a filter window \( W(K_x, K_y) \) which will eliminate the evanescent waves above a cut-off wavenumber \( K_c \).

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*Figure 1 - (a) Hologram plane and source plane; (b) Wavenumber space.*
Practical determination of the hologram

The theory of near-field acoustic holography (NAH) makes two assumptions:

(i) the hologram is large enough to represent the entire field originating from the source (and therefore infinite);

(ii) the pressure is known everywhere in the hologram.

In practice, the measurements are made using a microphone (or an intensity probe) over a mesh lattice with spatial increment \( \Delta l \). The well-known result of this sampling procedure is that the wavenumber spectrum becomes periodic (the period corresponds to the sampling wavenumber \( K_c = 2\pi / \Delta l \)). If the wavenumber spectrum of the field being analysed contains values greater than \( K_{max} = K_c / 2 = \pi / \Delta l \), the hologram will be under-sampled. Distortions due to overlapping spectra will artificially reinforce the evanescent components and seriously interfere with the reconstruction, because these distortions will be strongly amplified. It is therefore particularly important that Shannon’s sampling condition should be met for near-field acoustic holography owing to the backward propagation operation. However, the extent of the wavenumber spectrum of the field picked up by the hologram is not known at the outset and an initial data acquisition may sometimes result in the spatial increment \( \Delta l \) being modified.

Another source of distortion is introduced when the acquisition limits the size of the hologram recorded. In practice, the pressure field is non-zero outside the window of exploration, resulting in a truncating effect. If the total field possesses a limited wavenumber spectrum, that of the truncated field will no longer be so, resulting in an inevitable increase in evanescent waves owing to spectrum overlap.

Review of existing K-space windows

Existing filter windows are axisymmetrical in that \( W(K_x, K_y) = W(K_r) \) and the cut-off wave number is defined by \( W(K_c) = 0.5 \). A K-filter window published by Veronesi et al [4] is written in the following form

\[
W_1(K_x, K_y) = \begin{cases} 
1 - 0.5 \exp[(K_r/K_c - 1)/s], & \text{for } K_r \leq K_c, \\
0.5 \exp[-(K_r/K_c - 1)/s], & \text{for } K_r > K_c.
\end{cases}
\]  

(5)

![Figure 2](image_url)

Figure 2 – Comparison of filter windows: (i) use of Eq. (5) \( W_1(K_x, K_y) \); (ii) use of Eq. (6) \( W_2(K_x, K_y) \).

Both \( s \) and \( K_c \) are adjustable parameters which are proposed as \( s = 0.2 \), \( K_c = 0.6K_{max} \) (\( K_{max} = \pi N/L \)) with \( L \) the length of a square plane and \( N \) the number of mesh points. Figure 2 shows that this window function isn’t valid for the value \( s > 0.4 \). We therefore have modified it [7] to make it more regularly dependent on the form parameter \( s \):

\[
W_2(K_x, K_y) = \begin{cases} 
1, & \text{for } K_c = 0, \\
1 - 0.5 \exp[-(K_r/K_c - 1)/s], & \text{for } K_r \leq K_c, \\
0.5 \exp[-(K_r/K_c - 1)/s], & \text{for } K_r > K_c.
\end{cases}
\]  

(6)
To begin with, there are no criteria for choosing parameters, particularly as regards the cut-off wavenumber \( K_c \). However, Williams et al [1, 2] have stressed the relationship that exists between the signal to noise ratio \( S/B \) (expressed in dB), the distance of the hologram from the source and the limiting wavenumber \( K_l \) beyond which the evanescent components are no longer usable

\[
\frac{K_l}{k} \approx \frac{(S/B) \ln 10}{20 \, kd},
\]

(7)

with \( (S/B) = 10 \log_{10}(1/r_{bs}) \), where \( r_{bs} \) is the ratio of the energy of the noise to that of the signal. Other authors [8] have made use of this property to determine the window (Eq. 8) using a least squares minimisation method (LSM)

\[
W_3(K_x, K_y) = \begin{cases} 
1/(1 + r_{bs} K_r^2), & \text{for } K_r \leq k, \\
1/(1 + r_{bs} K_r^2 \exp[2d\sqrt{K_r^2 - k^2}]) , & \text{for } K_r > k.
\end{cases}
\]

(8)

![Figure 3 – Filter window \( W_3(K_x, K_y) \) (Eq. 8) shown as a function of the frequency for \( S/B = 30 \, \text{dB} \) and \( d = 0.09 \, \text{m} \).](image)

The filter window obtained no longer depends on arbitrary parameters but on factors defined by the experimental conditions, such as the noise to signal ratio \( r_{bs} \), the distance from the hologram to the source plane \( d \) and the frequency of analysis \( \omega \). This window is shown on Figure 3 as a function of frequency where \( r_{bs} = 0.001 \) \((S/B = 30 \, \text{dB})\) and \( d = 0.09 \, \text{m} \). It is clearly no longer valid when the frequency is high (high value of \( kd \)). In fact, the energy of the "signal" at the source plane considered for the calculation [8] is the sum of just those components in the kinetic energy plane \( T_{xy} = T_x + T_y = (\rho/4)\left(|u_x|^2 + |u_y|^2\right) \) \((u_x \text{ and } u_y \text{ are the components of the particular velocity vector along } x \text{ and } y\)). This approach to characterising the "signal" may seem arbitrary and debatable.

**DETERMINATION OF FILTERS BY AN OPTIMISATION METHOD**

The least squares method (LSM) involves minimising the square of the difference between the value \( p_M(x, y, z_H) \) obtained from measured data and its exact value \( p(x, y, z_H) \). The noise energy at the plane of the hologram can be expressed as a total potential energy \( \varepsilon = (1/4\rho c^2) \iint |p(x, y, z_H) - p_M(x, y, z_H)|^2 \, dx \, dy \) which can also be expressed in wavenumber space using the Parseval theorem thus

\[
\varepsilon = \frac{1}{4\rho c^2} \iint_{-\infty}^{+\infty} \iint_{-\infty}^{+\infty} |p(K_x, K_y, z_H) - p_M(K_x, K_y, z_H)|^2 \frac{dK_x}{2\pi} \frac{dK_y}{2\pi}.
\]

(10)
For optimal filtering of this noise, we try to minimise this quantity by giving an analytical expression to the "exact" value of the desired pressure \( p(K_x, K_y, z_H) = p(K_x, K_y, z_S) G_p(K_x, K_y, d) \). The total energy of the noise (Eq. 10) is then written

\[
\varepsilon = \frac{1}{4\rho c^2} \int \int \int \frac{+\infty+\infty}{-\infty-\infty} \left| p(K_x, K_y, z_S) G_p(K_x, K_y, d) - p_M(K_x, K_y, z_H) \right|^2 \frac{dK_x}{2\pi} \frac{dK_y}{2\pi}.
\] (11)

Potential energy condition

To produce a filtering window function, a further condition is added: the potential energy on the source plane must be finite, therefore

\[
\frac{1}{4\rho c^2} \int \int \int \frac{+\infty+\infty}{-\infty-\infty} \left| p(K_x, K_y, z_S) \right|^2 \frac{dK_x}{2\pi} \frac{dK_y}{2\pi} = \frac{1}{4\rho c^2} \int \int \int \frac{+\infty+\infty}{-\infty-\infty} \left| p(K_x, K_y, z_S) \right|^2 \frac{dK_x}{2\pi} \frac{dK_y}{2\pi} \leq E.
\] (12)

This condition is not satisfied if there is considered to be a uniform distribution of noise \( p_B(z_H) \) in wave number space at the hologram plane. In fact the integral (4) does not converge owing to the behaviour of \( G^\leftarrow_p(K_x, K_y, d) \) when \( K_\phi - k \). The desired filter will therefore have to modify this tendency. The noise to signal ratio defined above can be expressed here as \( r_{bs} = \varepsilon/E \), which allows Eq. (12) to be written

\[
\frac{1}{4\rho c^2} \int \int \int r_{bs} \left| p(K_x, K_y, z_S) \right|^2 \frac{dK_x}{2\pi} \frac{dK_y}{2\pi} \leq \varepsilon.
\] (13)

The problem is therefore to minimise the quantity

\[
Q = \frac{1}{4\rho c^2} \int \int \int \left[ \left| p(K_x, K_y, z_S) G_p(K_x, K_y, d) - p_M(K_x, K_y, z_H) \right|^2 + r_{bs} \left| p(K_x, K_y, z_S) \right|^2 \right] \frac{dK_x}{2\pi} \frac{dK_y}{2\pi}
\]

\[
= \frac{1}{4\rho c^2} \int \int \int F[K_x, K_y, p(K_x, K_y, z_S)] \frac{dK_x}{2\pi} \frac{dK_y}{2\pi},
\] (14)

which consists of two quadratic expressions (12) and (13) for the noise. Minimisation involves using the method of minimum variance which here involves resolving the equation

\[
\frac{\partial F[K_x, K_y, p(K_x, K_y, z_S)]}{\partial \text{Re}[p(K_x, K_y, z_S)]} + j \frac{\partial F[K_x, K_y, p(K_x, K_y, z_S)]}{\partial \text{Im}[p(K_x, K_y, z_S)]} = 0,
\] (15)

with respect to the pressure on the source plane. Finally we obtain

\[
p(K_x, K_y, z_S) = \frac{G^*_p(K_x, K_y, d) p_M(K_x, K_y, z_H)}{|G_p(K_x, K_y, d)|^2 + r_{bs}}.
\] (16)

This relationship is expressed in the form of Eq. (1) leading to

\[
W_4(K_x, K_y) = \begin{cases} 1/(1 + r_{sb}), & \text{for } K_\phi \leq k, \\ 1/(1 + r_{sb} \exp[2kd/\sqrt{(K^2_\phi/k^2)} - 1]), & \text{for } K_\phi > k. \end{cases}
\] (17)

It is the ratio \( r_{bs} \) which determines the cut-off wavenumber \( K_c \). To permit a comparison with the other filter windows, \( K_c \) is expressed using the definition given previously: \( W_4(K_c) = 0.5 \), whence

\[
\frac{K_c}{k} = \sqrt{\left[ \frac{(S/B)\ln 10}{20kd} \right]^2 + 1} \quad \text{with} \quad (S/B) = 10\log(1/r_{hs}).
\] (18)

This expression is to be compared with the limiting wavenumber obtained by Williams et al (Eq. 7): \( K_c/k \) tends to \( K_0/k \) when the first term of Eq. (18) is much greater than 1.
Total energy condition

In Eq. (13), the total potential energy is used as a criterion for estimating the energy of the "signal" on the source plane. However, in the near field where the normal component of reactive intensity is usually large, this quantity can fluctuate considerably in the direction $z$ [9]. It may therefore be interesting to replace it by an average of the potential energy $V$ and the kinetic energy $T$:

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{T(x, y, z_S) + V(x, y, z_S)}{2} \, dx \, dy = \frac{1}{8\rho c^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ \frac{\| \mathbf{\nabla} p(x, y, z_S) \|^2}{k^2} + \| p(x, y, z_S) \|^2 \right] \, dx \, dy \leq E. \quad (19)$$

Expressed in wavenumber space

$$\frac{1}{8\rho c^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ -j K_x \hat{1} + K_y \hat{j} + K_z \hat{k} \right] \frac{\left| p(K_x, K_y, z_S) \right|^2}{k} \, dK_x \, dK_y \leq E, \quad (20)$$

where $K_a = \sqrt{k^2 - K_z^2}$, this new condition replaces Eq. (13). It leads to the following:

$$\frac{1}{8\rho c^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left( 1 + \frac{K_x^2 + K_y^2 + K_z^2}{k^2} \right) \left| p(K_x, K_y, z_S) \right|^2 \frac{dK_x \, dK_y}{2\pi} \leq \varepsilon, \quad (21)$$

which will produce a new quantity $Q$ to be minimised. By following the same process as before, we obtain
a new expression for the filter window

$$W_5(K_x, K_y) = \begin{cases} \frac{1}{1 + r_{sb}}, & \text{for } K_r \leq k; \\ \frac{1}{1 + r_{sb} (K_r^2/k^2)} \exp[2kd\sqrt{(K_r^2/k^2) - 1}], & \text{for } K_r > k, \end{cases} \quad (22)$$

Comparison between K-space filters

When $K_r > k$, the filter window in respect of Eq. (22) is more narrow than the previous one (Eq. 17) for the same values of the parameters. Figure 4 shows the filter window $W_4$ (Eq. 17) and the filter window $W_5$ (Eq. 22) when $S/B = 20$ dB and $S/B = 30$ dB for different values of $kd$. The curves are plotted as a function of the reduced variable $K_r/k$. It will be noted (i) that the window $W_4$ is wider than window $W_5$ and that this phenomenon is particularly evident the smaller the value $kd$; (ii) the width and slope of $W_4$ and $W_5$ depend on the parameters $r_{bs}$ and $kd$. Naturally, the higher the signal to noise ratio $S/B$, the wider are the windows.

![Figure 5 - Illustration of filter windows $W_4$ (right) and $W_5$ (left) as a function of $K_r$ and frequency. The ratio $S/B$ and the distance $d$ are constant ($S/B = 30$ dB and $d = 0.09$ m).](image)

![Figure 6 - Correspondence between the filter windows $W_4$ and $W_5$ with the window $W_2$ at a frequency of 220 Hz (d = 0.09 m et S/B = 30 dB). (a) Parameters of $W_2$ to correspond to $W_4$ (left) : $s = 0.3$, $K_c = 38.6$ rad/m; (b) Parameters of $W_2$ to correspond to $W_5$ (right) : $s = 0.3$, $K_c = 21.5$ rad/m.](image)
On Figure 5, the two windows $W_4$ and $W_5$ are shown as a function of $K_r$. In this example, only the frequency has been varied in order to approach the conditions of a practical application where the signal to noise ratio, the distance $d$ and the maximum wavenumber remain constant. Figure 6 shows a comparison between the two windows $W_4$ and $W_5$ obtained by the least squares method and the parametric filter $W_2$ (Eq. 6). The parameters $s$ and $K_c$ of the filter window $W_2$ are adjusted to obtain the best correspondence with the windows $W_4$ and $W_5$. It will be noted that the LSM filter windows attenuate more for large values of $K_r$ than the window $W_2$, for the same cut-off wavenumbers.

CONCLUSION

The filtering of evanescent waves in wavenumber space raises a particularly delicate problem for the use of acoustic holography techniques. The approach used here, based upon the least squares method, can provide filtering solutions which are not based on an empirical definition of the filter windows, as has hitherto been the practice. Filtering of the wavenumber spectrum, at a given frequency, takes place only on the basis of the experimental parameters: the propagation distance and the signal to noise ratio. Two forms of filter were used, considering in one case a constraint on a finite potential energy at the source and, in the other case, a finite total energy (potential and kinetic). This second condition gives more satisfactory results [7]. It is also shown that a careful choice of parameters for the empirical filter most often used can approach the form of the optimal filter.

REFERENCES